

Few-anyon systems in a parabolic dot

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The energy levels of two and three anyons in a two-dimensional parabolic quantum dot and a perpendicular magnetic field are computed as power series in $1/|J|$, where J is the angular momentum. The particles interact repulsively through a coulombic ($1/r$) potential. In the two-anyon problem, the reached accuracy is better than one part in 10^5 . For three anyons, we study the combined effects of anyon statistics and coulomb repulsion in the “linear” anyonic states.

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I. INTRODUCTION

Recent developments in semiconductor technology (e.g. MBE and electron lithography) opened the possibility to create totally confined electron systems, the so called artificial atoms or quantum dots^{1,2}. This is one of the examples of present-day-physics’ interest in confined finite systems, among which one can mention also the atomic and electronic traps³, and the condensation of confined bosons⁴.

Quantum dots exhibit very interesting properties like the possibility of varying their parameters (number of electrons, applied fields, dot’s geometry, temperature) over a wide range, the observation of conductance oscillations⁵ and Kohn’s theorem⁶, etc.

Theoretically, these systems have been studied mainly with the help of numerical methods. However, analytic approaches have proven to work extremely well providing, at the same time, a qualitative understanding of the quantum dynamics. Among these approaches one can mention the semiclassical quantisation⁷, regularised perturbation theory^{8,9}, Pade approximant techniques^{10,11}, and the $1/N$ -expansion^{12–14}.

In the present paper, we continue the analytic-qualitative line of research and apply the $1/N$ -expansion (N is the absolute value of the angular momentum) to compute the energy levels of two and three anyons in a model parabolic dot. The particles interact through a coulombic ($1/r$) repulsive potential. A magnetic field is applied perpendicularly to the plane of motion.

Numerical results for the two-anyon system were obtained by Myrheim et al¹⁵. Exact analytic solutions at particular values of the coupling constants were found in¹⁶. Bohr-Sommerfeld quantisation was applied to this system at low magnetic fields¹⁷. We shall see that our method provides extremely accurate solutions for states

with angular momentum $|J| \geq 2$. A picture for the “geometry” of the states (the spatial distribution of probability) is obtained also. In the three-anyon problem, however, to our knowledge there are no numerical or approximate calculations.

We start from the Hamiltonian of N_a anyons moving in a two dimensional quantum dot in the presence of a perpendicular homogenous magnetic field. In the bosonic gauge, it is given by the expression¹⁸

$$H = \frac{1}{2m} \sum_{i=1}^{N_a} \left| \vec{p}_i + \frac{e}{c} \vec{A}_i - \hbar \nu \vec{a}_i \right|^2 + \frac{m}{2} \omega_0^2 \sum_{i=1}^{N_a} r_i^2 + \sum_{i>j} \frac{e^2}{\epsilon |\vec{r}_i - \vec{r}_j|}, \quad (1)$$

in which the vector potential is taken in the symmetric gauge,

$$\vec{A}_i = \frac{1}{2} \vec{B} \times \vec{r}_i, \quad (2)$$

\vec{a}_i is the statistical vector potential,

$$\vec{a}_i = \sum_{j \neq i} \frac{\vec{n} \times (\vec{r}_j - \vec{r}_i)}{|\vec{r}_j - \vec{r}_i|^2}, \quad (3)$$

\vec{n} is the unit vector perpendicular to the plane of motion of the anyons, e is the anyon’s charge, ν is the anyonic parameter, ω_0 is the frequency of the parabolic potential needed to confine the anyons in the dot and ϵ is the dielectric constant of the medium. A dimensionless Hamiltonian is obtained by means of the change of variables $\vec{r}_i \rightarrow \sqrt{\hbar/(m\Omega)} \vec{r}_i$

$$\begin{aligned} \frac{H}{\hbar\Omega} = & \frac{1}{2} \sum_{i=1}^{N_a} p_i^2 + \frac{\omega_c}{2\Omega} \vec{n} \cdot \sum_{i=1}^{N_a} \vec{r}_i \times \vec{p}_i \\ & + \nu \vec{n} \cdot \sum_{i>j} \frac{(\vec{r}_i - \vec{r}_j) \times (\vec{p}_i - \vec{p}_j)}{|\vec{r}_i - \vec{r}_j|^2} \\ & + \frac{1}{2} \sum_{i=1}^{N_a} r_i^2 + \frac{\omega_c}{4\Omega} \nu N_a (N_a - 1) \\ & + \frac{\nu^2}{2} \sum_{i \neq j, k} \frac{(\vec{r}_i - \vec{r}_j) \cdot (\vec{r}_i - \vec{r}_k)}{|\vec{r}_i - \vec{r}_j|^2 |\vec{r}_i - \vec{r}_k|^2} + \beta^3 \sum_{i>j} \frac{1}{|\vec{r}_i - \vec{r}_j|}, \quad (4) \end{aligned}$$

where $\omega_c = eB/(mc)$ is the cyclotronic frequency, $\Omega^2 = \omega_c^2/4 + \omega_0^2$, and $\beta^3 = \sqrt{me^4/(\epsilon^2 \Omega \hbar^3)}$ is the square root

of the ratio between the coulombic and oscillator characteristic energies. The problem has one exactly solvable limit: a low-density limit, which we call the Wigner limit, reached when $\beta \rightarrow \infty$. In the $\beta \rightarrow 0$ (oscillator) limit, the two-anyon problem is exactly solvable¹⁹, whereas the three-anyon system has an infinite family of exact linear states²⁰. In real semiconductors, $\beta \sim 1$.

Introducing Jacobi coordinates,

$$\vec{\rho}_k = \sqrt{\frac{2k}{k+1}} \left\{ \frac{1}{k} \sum_{i=1}^k \vec{r}_i - \vec{r}_{k+1} \right\}, \quad 1 \leq k \leq N_a - 1, \quad (5)$$

$$\vec{\rho}_{N_a} = \frac{1}{\sqrt{N_a}} \sum_{i=1}^{N_a} \vec{r}_i, \quad (6)$$

the centre-of-mass and relative motions are separated

$$\frac{H}{\hbar\Omega} = \frac{H_{CM}}{\hbar\Omega} + \frac{H_{rel}}{\hbar\Omega}, \quad (7)$$

where

$$\frac{H_{CM}}{\hbar\Omega} = \frac{1}{2} p_{N_a}^2 + \frac{\omega_c}{2\Omega} \vec{n} \cdot (\vec{\rho}_{N_a} \times \vec{p}_{N_a}) + \frac{1}{2} \rho_{N_a}^2, \quad (8)$$

is the centre of mass Hamiltonian and

$$\begin{aligned} \frac{H_{rel}}{\hbar\Omega} = & \sum_{i=1}^{N_a-1} p_i^2 + \frac{\omega_c}{2\Omega} \vec{n} \cdot \sum_{i=1}^{N_a-1} \vec{\rho}_i \times \vec{p}_i \\ & + \nu \vec{n} \cdot \sum_{i>j} \frac{\vec{r}_{ij} \times \vec{p}_{ij}}{r_{ij}^2} + \frac{1}{4} \sum_{i=1}^{N_a-1} \rho_i^2 + \frac{\omega_c}{4\Omega} \nu N_a (N_a - 1) \\ & + \frac{\nu^2}{2} \sum_{i \neq j, k} \frac{\vec{r}_{ij} \cdot \vec{r}_{ik}}{r_{ij}^2 r_{ik}^2} + \beta^3 \sum_{i>j} \frac{1}{r_{ij}}, \end{aligned} \quad (9)$$

is the Hamiltonian of the relative motion. We introduced the following notation: $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$, and $\vec{p}_{ij} = \vec{p}_i - \vec{p}_j$.

We will obtain approximate expressions for the energy eigenvalues of $H_{rel}/(\hbar\Omega)$ for $N_a = 2$ and $N_a = 3$ as a function of β by means of the $1/|J|$ -expansion.

II. THE TWO-ANYON SYSTEM

In the two-anyon problem, we have only one Jacobi coordinate, $\vec{\rho}_1$, and the Hamiltonian of the internal motion reads

$$\begin{aligned} \frac{H_{rel}}{\hbar\Omega} = & p_1^2 + \frac{\omega_c}{2\Omega} \vec{n} \cdot (\vec{\rho}_1 \times \vec{p}_1) + 2\nu \vec{n} \cdot \frac{\vec{\rho}_1 \times \vec{p}_1}{\rho_1^2} + \frac{1}{4} \rho_1^2 \\ & + \frac{\nu^2}{\rho_1^2} + \frac{\beta^3}{\rho_1} + \frac{\omega_c \nu}{2\Omega}. \end{aligned} \quad (10)$$

Notice that $\vec{n} \cdot (\vec{\rho}_1 \times \vec{p}_1) = J$. After the scaling transformation $\rho_1^2 \rightarrow |J| \rho_1^2$, we get

$$\begin{aligned} h = & \frac{1}{|J|} \left[\frac{H_{rel}}{\hbar\Omega} - \frac{\omega_c(J + \nu)}{2\Omega} \right] = \frac{(\tilde{\nu} + 1)^2}{\rho_1^2} + \frac{1}{4} \rho_1^2 + \frac{\tilde{\beta}^3}{\rho_1} \\ & - \frac{1}{J^2} \left(\frac{\partial^2}{\partial \rho_1^2} + \frac{1}{\rho_1} \frac{\partial}{\partial \rho_1} \right), \end{aligned} \quad (11)$$

where $\rho_1 = |\vec{\rho}_1|$. We have “renormalised” the coupling constant, $\tilde{\beta}^3 = \beta^3/|J|^{\frac{3}{2}}$, and the statistical parameter, $\tilde{\nu} = \nu/J$, in order to take account of the Coulomb repulsion and the statistical interaction in a nonperturbative way when taking the formal limit $|J| \rightarrow \infty$. We shall look for symmetric eigenfunctions of h , i.e. $|J|$ shall be even.

In the $|J| \rightarrow \infty$ limit, the only term surviving in the Hamiltonian is the effective (classical) potential energy

$$U_{eff} = \frac{(\tilde{\nu} + 1)^2}{\rho_1^2} + \frac{1}{4} \rho_1^2 + \frac{\tilde{\beta}^3}{\rho_1}. \quad (12)$$

Minimising U_{eff} , we obtain the leading contribution to the energy, $\epsilon_0 = U_{eff}(\rho_{01})$, where the radius of the “Bohr orbit” is obtained from

$$\frac{1}{2} \rho_{01}^4 - \tilde{\beta}^3 \rho_{01} = 2(\tilde{\nu} + 1)^2. \quad (13)$$

Substituting $\rho_1 = \rho_{01} + y_1/|J|^{1/2}$ in the r.h.s. of (11) and expanding, we get

$$h = \sum_{i=0}^{\infty} \frac{h_i}{|J|^{i/2}}, \quad (14)$$

where the operator coefficients are given by

$$h_0 = \frac{3}{4} \rho_{01}^2 - \frac{(\tilde{\nu} + 1)^2}{\rho_{01}^2}, \quad (15)$$

$$h_1 = 0, \quad (16)$$

$$h_2 = -\frac{\partial^2}{\partial y_1^2} + \frac{1}{4} \left(3 + \frac{4(\tilde{\nu} + 1)^2}{\rho_{01}^4} \right) y_1^2 \quad (17)$$

$$\begin{aligned} h_i = & (-1)^i \left\{ \left(\frac{1}{2\rho_{01}^{i-2}} + \frac{(i-1)(\tilde{\nu} + 1)^2}{\rho_{01}^{i+2}} \right) y_1^i \right. \\ & \left. + \frac{1}{\rho_{01}^{i-2}} y_1^{i-3} \frac{\partial}{\partial y_1} \right\}, \quad i \geq 3. \end{aligned} \quad (18)$$

Similar series are written for the wave function and the scaled energy, that is,

$$\psi = \sum_{i=0}^{\infty} \frac{\psi_i}{|J|^{i/2}}, \quad (19)$$

$$\epsilon = \frac{1}{|J|} \left[\frac{E_{rel}}{\hbar\Omega} - \frac{\omega_c(J + \nu)}{2\Omega} \right] = \sum_{i=0}^{\infty} \frac{\epsilon_i}{|J|^{i/2}}. \quad (20)$$

Inserting (19), (20) and the series expansion for the Hamiltonian into the Schrödinger equation, we may compute the coefficients ψ_i and ϵ_i in a systematic way. Up to second order, for example, the system is described in

terms of small oscillations around the equilibrium orbit, i.e. the wave function is

$$\Psi_0 = e^{iJ\theta} |n\rangle, \quad (21)$$

where θ is the angle associated to the vector $\vec{\rho}_1$, the $|n\rangle$ are two-dimensional harmonic oscillator radial states with frequency $\omega_1 = \sqrt{3 + 4(\tilde{\nu} + 1)^2/\rho_{01}^4}$, and the first two coefficients for the energy are

$$\epsilon_0 = \frac{3}{4}\rho_{01}^2 - \frac{(\tilde{\nu} + 1)^2}{\rho_{01}^2}, \quad (22)$$

$$\epsilon_2 = \omega_1 \left(n + \frac{1}{2} \right). \quad (23)$$

Afterward, we may take account of anharmonicities. The results for the next two coefficients are the following

$$\epsilon_4 = -\frac{1}{4\rho_{01}^2} + \frac{3(2n^2 + 2n + 1)}{2\omega_1^2\rho_{01}^2} \left(1 + \frac{6(\tilde{\nu} + 1)^2}{\rho_{01}^4} \right) - \frac{(30n^2 + 30n + 11)}{4\omega_1^4\rho_{01}^2} \left(1 + \frac{4(\tilde{\nu} + 1)^2}{\rho_{01}^4} \right)^2, \quad (24)$$

$$\begin{aligned} \epsilon_6 = & -\frac{3(2n + 1)}{4\omega_1\rho_{01}^4} + \frac{(2n + 1)}{\omega_1^3\rho_{01}^4} [5n^2 + 5n + 9 \\ & + \frac{(\tilde{\nu} + 1)^2(50n^2 + 50n + 81)}{\rho_{01}^4}] - \frac{(2n + 1)}{2\omega_1^5\rho_{01}^4} [87n^2 \\ & + 87n + 86 + \frac{12(\tilde{\nu} + 1)^2(87n^2 + 87n + 86)}{\rho_{01}^4} \\ & + \frac{4(\tilde{\nu} + 1)^4(713n^2 + 713n + 709)}{\rho_{01}^8}] + \frac{9(2n + 1)}{2\omega_1^7\rho_{01}^4} \\ & \times (25n^2 + 25n + 19) \left(1 + \frac{6(\tilde{\nu} + 1)^2}{\rho_{01}^4} \right) \\ & \times \left(1 + \frac{4(\tilde{\nu} + 1)^2}{\rho_{01}^4} \right)^2 - \frac{15(2n + 1)}{8\omega_1^9\rho_{01}^4} (47n^2 + 47n + 31) \\ & \times \left(1 + \frac{4(\tilde{\nu} + 1)^2}{\rho_{01}^4} \right)^4. \end{aligned} \quad (25)$$

Notice that in both Wigner ($\beta \rightarrow \infty$) and oscillator ($\beta \rightarrow 0$) limits the corrections ϵ_4 and ϵ_6 go to zero. The expressions found in¹⁴ are reproduced if we take $\nu = 0$.

We show in Fig. 1 the relative weight of ϵ_6 in ϵ for the first states with $J = 2$ and 6. The parameter ν was fixed to 1/2 (semions). The relative contribution of ϵ_6 is never greater than 5×10^{-5} or 3×10^{-6} for $J = 2$ or 6 respectively. This shows that the $1/|J|$ -series is extremely well behaved.

A comparison with the exact solutions found in¹⁶ is carried on in Fig. 2, where the relative difference $|\epsilon - \epsilon_{exact}|/\epsilon_{exact}$ is plotted against ν . The state with $J = 6$, $n = 0$ is shown. It may be easily verified that $\psi_{exact} = \rho_1^{|J+\nu|} (1 + \rho_1/\sqrt{2|J+\nu|+1}) e^{-\rho_1^2}$, $\epsilon_{exact} = |J+\nu| + 2$, are exact solutions of the two-anyon problem at $\beta^3 = \sqrt{2|J+\nu|+1}$. The comparison shows that the relative error of our estimate is not greater than 10^{-8} .

III. THE THREE-ANYON SYSTEM

The internal Hamiltonian of the system of three anyons in Jacobi coordinates $\vec{\rho}_1$ and $\vec{\rho}_2$ is written as

$$\begin{aligned} \frac{H_{rel}}{\hbar\Omega} = & \sum_{i=1}^2 p_i^2 + \frac{\omega_c}{2\Omega} \vec{n} \cdot \sum_{i=1}^2 \vec{\rho}_i \times \vec{p}_i + \frac{3\omega_c\nu}{2\Omega} \\ & + \nu \vec{n} \cdot \left[2 \frac{\vec{\rho}_1 \times \vec{p}_1}{\rho_1^2} + 2 \frac{(\vec{\rho}_1 + \sqrt{3}\vec{\rho}_2) \times (\vec{p}_1 + \sqrt{3}\vec{p}_2)}{|\vec{\rho}_1 + \sqrt{3}\vec{\rho}_2|^2} \right. \\ & \left. + 2 \frac{(\vec{\rho}_1 - \sqrt{3}\vec{\rho}_2) \times (\vec{p}_1 - \sqrt{3}\vec{p}_2)}{|\vec{\rho}_1 - \sqrt{3}\vec{\rho}_2|^2} \right] + \frac{1}{4} \sum_{i=1}^2 \rho_i^2 \\ & + 9\nu^2 \frac{(\rho_1^2 + \rho_2^2)^2 + 4(\rho_1^2\rho_2^2 - (\vec{\rho}_1 \cdot \vec{\rho}_2)^2)}{\rho_1^2 |\vec{\rho}_1 + \sqrt{3}\vec{\rho}_2|^2 |\vec{\rho}_1 - \sqrt{3}\vec{\rho}_2|^2} + \\ & \beta^3 \left[\frac{1}{\rho_1} + \frac{2}{|\vec{\rho}_1 + \sqrt{3}\vec{\rho}_2|} + \frac{2}{|\vec{\rho}_1 - \sqrt{3}\vec{\rho}_2|} \right]. \end{aligned} \quad (26)$$

Doing the same scaling transformation $\rho_i^2 \rightarrow |J|\rho_i^2$, and making explicit the dependence on $|J|$, we get

$$\begin{aligned} \frac{1}{|J|} \left[\frac{H_{rel}}{\hbar\Omega} - \frac{\omega_c(J+3\nu)}{2\Omega} \right] = & \frac{1}{4} \left(\frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} \right) + \frac{1}{4} (\rho_1^2 + \rho_2^2) \\ & + \tilde{\beta}^3 \left[\frac{1}{\rho_1} + \frac{2}{|\vec{\rho}_1 + \sqrt{3}\vec{\rho}_2|} + \frac{2}{|\vec{\rho}_1 - \sqrt{3}\vec{\rho}_2|} \right] \\ & + 4\tilde{\nu} \frac{\rho_1^2(2 - 3\cos^2\theta) + 3\rho_2^2(2 - \cos^2\theta)}{|\vec{\rho}_1 + \sqrt{3}\vec{\rho}_2|^2 |\vec{\rho}_1 - \sqrt{3}\vec{\rho}_2|^2} + \frac{\tilde{\nu}}{\rho_1^2} \\ & + 9\tilde{\nu}^2 \frac{(\rho_1^2 + \rho_2^2)^2 + 4\rho_1^2\rho_2^2\sin^2\theta}{\rho_1^2 |\vec{\rho}_1 + \sqrt{3}\vec{\rho}_2|^2 |\vec{\rho}_1 - \sqrt{3}\vec{\rho}_2|^2} + \frac{1}{J} \left[\left(\frac{1}{\rho_1^2} - \frac{1}{\rho_2^2} \right) \frac{\partial}{\partial\theta} \right. \\ & + \tilde{\nu} \left(12i \frac{\rho_1\rho_2\sin 2\theta}{|\vec{\rho}_1 + \sqrt{3}\vec{\rho}_2|^2 |\vec{\rho}_1 - \sqrt{3}\vec{\rho}_2|^2} \left(\rho_1 \frac{\partial}{\partial\rho_2} - \rho_2 \frac{\partial}{\partial\rho_1} \right) \right. \\ & \left. \left. + i \frac{2}{\rho_1^2} \frac{\partial}{\partial\theta} - 8i \frac{\rho_1^2(1 - 3\cos^2\theta) + 3\rho_2^2(1 + \cos^2\theta)}{|\vec{\rho}_1 + \sqrt{3}\vec{\rho}_2|^2 |\vec{\rho}_1 - \sqrt{3}\vec{\rho}_2|^2} \frac{\partial}{\partial\theta} \right) \right] \\ & + \frac{1}{J^2} \left[- \left(\frac{\partial^2}{\partial\rho_1^2} + \frac{1}{\rho_1} \frac{\partial}{\partial\rho_1} + \frac{\partial^2}{\partial\rho_2^2} + \frac{1}{\rho_2} \frac{\partial}{\partial\rho_2} + \right. \right. \\ & \left. \left. \left(\frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} \right) \frac{\partial^2}{\partial\theta^2} \right) \right], \end{aligned} \quad (27)$$

where $\rho_1 = |\vec{\rho}_1|$, $\rho_2 = |\vec{\rho}_2|$, and $\cos\theta = \vec{\rho}_1 \cdot \vec{\rho}_2/(\rho_1\rho_2)$. The “renormalised” $\tilde{\beta}^3 = \beta^3/|J|^{\frac{3}{2}}$ and $\tilde{\nu} = \nu/J$ were introduced.

The minimum of the classical potential entering (27) is reached in the configuration of an equilateral triangle ($\rho_{01} = \rho_{02}$, $\theta = \pm\pi/2$). We choose, for example, $\rho_{01} = \rho_{02}$, $\theta = \pi/2$. ρ_{01} is obtained as the solution of the equation

$$\rho_{01}^4 - 3\rho_{01}\tilde{\beta}^3 = (3\tilde{\nu} + 1)^2. \quad (28)$$

Then, introducing $\rho_1 = \rho_{01} + y_1/\sqrt{|J|}$, $\rho_2 = \rho_{01} + y_2/\sqrt{|J|}$ and $\theta = \pi/2 + z/\sqrt{|J|}$ in the r.h.s of (27), we obtain for the Hamiltonian h a series like (14). The first operator coefficients are given by

$$h_0 = \frac{3}{2}\rho_{01}^2 - \frac{(3\tilde{\nu}+1)^2}{2\rho_{01}^2}, \quad (29)$$

$$h_1 = 0, \quad (30)$$

$$\begin{aligned} h_2 = & - \left(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} + \frac{2}{\rho_{01}^2} \frac{\partial^2}{\partial z^2} \right) \\ & - \frac{2i}{\rho_{01}^3} \text{sign}(J) (y_1 - y_2) \frac{\partial}{\partial z} + \frac{1}{4} \left(\frac{3}{\rho_{01}^4} + 1 \right) (y_1^2 + y_2^2) \\ & + \frac{1}{16} \left(1 - \frac{1}{\rho_{01}^4} \right) (5y_1^2 + 6y_1y_2 + 5y_2^2 + 3\rho_{01}^2 z^2) \\ & + \frac{9\tilde{\nu}^2}{16\rho_{01}^4} (y_1^2 + y_2^2 + 6y_1y_2 - \rho_{01}^2 z^2) \\ & + \tilde{\nu} \left[\frac{3}{8\rho_{01}^4} (3y_1^2 + 3y_2^2 + 2y_1y_2) - \frac{9z^2}{8\rho_{01}^2} \right. \\ & + \frac{3i \text{sign}(J)z}{2\rho_{01}} \left(\frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2} \right) \\ & \left. - \frac{3i \text{sign}(J)}{\rho_{01}^3} (y_1 - y_2) \frac{\partial}{\partial z} \right], \quad (31) \end{aligned}$$

$$\begin{aligned} h_3 = & - \frac{1}{\rho_{01}} \left(\frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} \right) + \frac{2}{\rho_{01}^3} (y_1 + y_2) \\ & + \frac{3i \text{sign}(J)}{\rho_{01}^4} (y_1^2 - y_2^2) \frac{\partial}{\partial z} - \frac{1}{\rho_{01}^5} (y_1^3 + y_2^3) \\ & - \frac{1}{64\rho_{01}} \left(1 - \frac{1}{\rho_{01}^4} \right) (19y_1^3 + 3y_1^2y_2 + 33y_1y_2^2 + 9y_2^3 \\ & - 9\rho_{01}^2 y_1 z^2 + 21\rho_{01}^2 y_2 z^2) - \frac{9\tilde{\nu}^2}{64\rho_{01}^5} (y_2^3 - 5y_1^3 - 21y_1^2y_2 \\ & - 39y_1y_2^2 + 7\rho_{01}^2 y_1 z^2 - 11\rho_{01}^2 y_2 z^2) \\ & + \tilde{\nu} \left[\frac{3i \text{sign}(J)}{2\rho_{01}^4} \left\{ (4y_1^2 - 2y_2^2 - 2y_1y_2 - \rho_{01}^2 z^2) \frac{\partial}{\partial z} \right. \right. \\ & \left. \left. - \rho_{01}^2 z \left(y_1 \frac{\partial}{\partial y_2} + y_2 \frac{\partial}{\partial y_1} \right) + 2\rho_{01}^2 y_2 z \frac{\partial}{\partial y_2} \right\} \right. \\ & \left. - \frac{3}{32\rho_{01}^5} (21y_1^3 + 15y_2^3 + 5y_1^2y_2 + 23y_1y_2^2 \right. \\ & \left. + \rho_{01}^2 y_1 z^2 - 13\rho_{01}^2 y_2 z^2) \right], \quad (32) \end{aligned}$$

$$\begin{aligned} h_4 = & \frac{1}{\rho_{01}^2} \left(y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2} \right) - \frac{3}{\rho_{01}^4} (y_1^2 + y_2^2) \frac{\partial^2}{\partial z^2} \\ & - \frac{4i \text{sign}(J)}{\rho_{01}^5} (y_1^3 - y_2^3) \frac{\partial}{\partial z} + \frac{5}{4\rho_{01}^6} (y_1^4 + y_2^4) \\ & + \frac{1}{256\rho_{01}^2} \left(1 - \frac{1}{\rho_{01}^4} \right) \left(\frac{329}{4} y_1^4 - 25y_1^3y_2 \right. \\ & + \frac{123}{2} y_1^2y_2^2 + 135y_1y_2^3 + \frac{9}{4} y_2^4 - \frac{99}{2} \rho_{01}^2 y_1^2 z^2 \\ & \left. + 27\rho_{01}^2 y_1y_2 z^2 + \frac{141}{2} \rho_{01}^2 y_2^2 z^2 + \frac{41}{4} \rho_{01}^4 z^4 \right) \end{aligned}$$

$$\begin{aligned} & + \tilde{\nu} \left[- \frac{i \text{sign}(J)}{8\rho_{01}^5} \left\{ 6 (11y_1^3 - y_1^2y_2 - 7y_1y_2^2 - 3y_2^3 \right. \right. \\ & - 3\rho_{01}^2 y_1 z^2 - \rho_{01}^2 y_2 z^2) \frac{\partial}{\partial z} \\ & + \rho_{01}^2 z (9y_1^2 - 3y_2^2 - 18y_1y_2 - \rho_{01}^2 z^2) \frac{\partial}{\partial y_1} \\ & \left. - \rho_{01}^2 z (9y_1^2 - 27y_2^2 + 6y_1y_2 - \rho_{01}^2 z^2) \frac{\partial}{\partial y_2} \right\} \\ & + \frac{3}{512\rho_{01}^6} (503y_1^4 - 28y_1^3y_2 + 266y_1^2y_2^2 + 612y_1y_2^3 \\ & + 183y_2^4 - 58\rho_{01}^2 y_1^2 z^2 + 20\rho_{01}^2 y_1y_2 z^2 - 154\rho_{01}^2 y_2^2 z^2 \\ & - 41\rho_{01}^4 z^4) + \frac{3\tilde{\nu}^2}{1024\rho_{01}^6} (549y_1^4 - 411y_2^4 - 84y_1^3y_2 \\ & + 2718y_1^2y_2^2 + 1836y_1y_2^3 - 750\rho_{01}^2 y_1^2 z^2 + 1266\rho_{01}^2 y_2^2 z^2 \\ & + 60\rho_{01}^2 y_1y_2 z^2 + 37\rho_{01}^4 z^4) \}. \quad (33) \end{aligned}$$

The operator h_2 is diagonalised by changing variables $y_1 = (y_s + y_m)/\sqrt{2}$, $y_2 = (y_s - y_m)/\sqrt{2}$, $z = \sqrt{2}z_m/\rho_{01}$, and making the “gauge” transformation $h'_2 = e^{if} h_2 e^{-if}$ where $f = \text{sign}(J)y_m z_m/(2\rho_{01}^2)$, the results is

$$h'_2 = h_s + h_m, \quad (34)$$

where h_s describes the motion of a harmonic oscillator in the coordinate y_s (the symmetric mode),

$$h_s = - \frac{\partial^2}{\partial y_s^2} + \frac{\omega_1^2}{4} y_s^2, \quad (35)$$

with $\omega_1 = \sqrt{3 + (3\tilde{\nu} + 1)^2/\rho_{01}^4}$, and h_m accounts for two dimensional motion in a “fictitious” magnetic field (the mixed mode)

$$\begin{aligned} h_m = & - \frac{\partial^2}{\partial y_m^2} - \frac{\partial^2}{\partial z_m^2} + \frac{\omega_2^2}{4} (y_m^2 + z_m^2) \\ & - \frac{i \text{sign}(J)(3\tilde{\nu} + 1)}{\rho_{01}^2} \left(y_m \frac{\partial}{\partial z_m} - z_m \frac{\partial}{\partial y_m} \right), \quad (36) \end{aligned}$$

where $\omega_2 = \sqrt{3/2 - (3\tilde{\nu} + 1)^2/(2\rho_{01}^4)}$.

The first two coefficients in the expansion for the energy are

$$\epsilon_0 = \frac{3}{2}\rho_{01}^2 - \frac{(3\tilde{\nu}+1)^2}{2\rho_{01}^2}, \quad (37)$$

$$\epsilon_2 = \omega_1 \left(n_s + \frac{1}{2} \right) + \omega_2 (2n + |m| + 1) + \omega_3 \text{sign}(J)m, \quad (38)$$

where $\omega_3 = (3\tilde{\nu} + 1)/\rho_{01}^2$, and the quantum numbers n_s , n , and m may be used to approximately label the states.

Up to this order, the wave function is given by

$$\Psi_0 = e^{iJ\Xi} |J, n_s, n, m\rangle, \quad (39)$$

where Ξ accounts for overall rotations of the system, and $|J, n_s, n, m\rangle$ are the eigenfunctions of h'_2 . We notice that, when $\beta \rightarrow 0$ the energy becomes

$$E_0 = |J + 3\nu| + 2 + 2n_s + 2n + |m| + \text{sign}(J)m. \quad (40)$$

These are the “linear” three-anyon states. We stress that they are obtained as harmonic excitations against the equilateral triangle configuration, and are not necessarily related to a cigar-like shape of the wave function ($\rho_1 \gg \rho_2$) as described in²¹.

The set of numbers $\{J, n_s, n, m\}$ compatible with the symmetry constraints (the wave function shall be symmetric) are obtained upon comparison with harmonic-oscillator wave functions at $\nu = 0, \beta = 0$. Details may be found in^{22,14}. An additional requirement is that the state at $\beta = 0$ should be a linear state. For example, the lowest linear states are the following: $|0, 0, 0, 0\rangle$ (the g.s., starting from $E_0 = 2$ at the bosonic end), $|0, 1, 0, 0\rangle$, and $|2, 0, 0, -1\rangle$ (starting from $E_0 = 4$), $|3, 0, 0, 0\rangle$ and $|1, 0, 0, 1\rangle$ (starting from $E_0 = 5$), $|4, 0, 0, -2\rangle$, $|2, 1, 0, -1\rangle$, and $|0, 1, 1, 0\rangle$ (starting from $E_0 = 6$), etc. The lowest state with $J < 0$ is $|-6, 0, 0, 0\rangle$, which starts at $E_0 = 8$. Of course, states with small values of $|J|$ can not be described within our method.

In what follows, we restrict the analysis to levels with quantum numbers $n_s = n = m = 0$. This leaves only the linear anyonic states with $J = 3k$, where k is an integer²². The geometry of the state is an equilateral triangle. It can be seen from (29) that the side of the triangle increases with ν when $J > 0$, and decreases when $J < 0$. Thus, the coulomb repulsion is much more stronger for $J < 0$ states, and the ordering of levels may dramatically change as β is increased. On the other hand, for $\beta \rightarrow \infty$ the side grows like $\rho_{01} \sim 3^{1/3}\beta$ and becomes independent of the anyonic parameter, as one expects. A strong coupling expansion¹⁰ shows that the leading contribution to the energy (potential energy) is $\sim \beta^2$, the next corrections (quantum fluctuations) are ~ 1 , the angular momentum and the statistical parameter enter the second order corrections, which are $\sim 1/\beta^2$.

The first anharmonic corrections to the energy are given by

$$\begin{aligned} \epsilon_4 = & \langle J, n_s, n, m | h'_4 | J, n_s, n, m \rangle \\ & + \sum_{n'_s, n', m'} \frac{\langle J, n_s, n, m | h'_3 | J, n'_s, n', m' \rangle}{\epsilon_2^{J n_s n m} - \epsilon_2^{J n'_s n' m'}} \\ & \times \langle J, n'_s, n', m' | h'_3 | J, n_s, n, m \rangle, \end{aligned} \quad (41)$$

where h'_3 and h'_4 are obtained from h_3 and h_4 by means of a gauge transformation, in the same way as explained above for h_2 .

For a state with quantum numbers $|J, 0, 0, 0\rangle$, we get

$$\begin{aligned} & \langle J, 0, 0, 0 | h'_4 | J, 0, 0, 0 \rangle = \\ & -\frac{5}{8\rho_{01}^2} + \frac{3}{4}\frac{1}{\rho_{01}^2\omega_1^2} + \frac{9}{8}\frac{1}{\rho_{01}^6\omega_1^2} + \frac{9}{16}\frac{1}{\rho_{01}^2\omega_1\omega_2} \left(1 + \frac{1}{\rho_{01}^4}\right) \\ & + \frac{3}{8}\frac{\omega_2}{\rho_{01}^2\omega_1} + \frac{9}{16}\frac{1}{\rho_{01}^2\omega_1^2} \left(1 - \frac{1}{\rho_{01}^4}\right) \\ & + \frac{27\tilde{\nu}^2}{16\rho_{01}^6\omega_1^2\omega_2^2} (3\omega_1^2 - \omega_1\omega_2 + 6\omega_2^2) \end{aligned}$$

$$- \frac{3\tilde{\nu}}{8\rho_{01}^6\omega_1^2\omega_2^2} (9\omega_1^2 - \omega_1\omega_2 - 18\omega_2^2), \quad (42)$$

$$h'_3 = A \frac{\partial}{\partial y_s} + B y_s + C y_s^3 + D, \quad (43)$$

where

$$A = -\frac{\sqrt{2}}{\rho_{01}} - \frac{3\sqrt{2}\iota \text{sign}(J)\tilde{\nu}}{4\rho_{01}^3} \xi^2 \sin 2\alpha, \quad (44)$$

$$\begin{aligned} B = & \frac{\sqrt{2}}{\rho_{01}} \left\{ \sin^2 \alpha \frac{\partial^2}{\partial \xi^2} + \frac{\cos^2 \alpha}{\xi^2} \frac{\partial^2}{\partial \alpha^2} \right. \\ & + \left(\frac{(3\tilde{\nu} + 2)\iota \text{sign}(J)}{\rho_{01}^2} \cos^2 \alpha - \frac{\sin 2\alpha}{\xi^2} \right. \\ & + \left. \left. \frac{3\tilde{\nu}\iota \text{sign}(J)}{2\rho_{01}^2} \right) \frac{\partial}{\partial \alpha} \right. \\ & + \left(\frac{(3\tilde{\nu} + 2)\iota \text{sign}(J) \sin 2\alpha \xi}{2\rho_{01}^2} + \frac{\cos^2 \alpha}{\xi} \right) \frac{\partial}{\partial \xi} \\ & + \frac{\sin 2\alpha}{\xi} \frac{\partial^2}{\partial \xi \partial \alpha} + \frac{9\tilde{\nu}^2 - 1}{4\rho_{01}^4} \xi^2 \cos^2 \alpha \\ & \left. - \frac{3}{16} \left(1 + \frac{3\tilde{\nu}^2 - 2\tilde{\nu} - 1}{\rho_{01}^4} \right) \xi^2 \right\}, \end{aligned} \quad (45)$$

$$C = -\frac{\sqrt{2}}{4\rho_{01}} \left(1 + \frac{(3\tilde{\nu} + 1)^2}{\rho_{01}^4} \right), \quad (46)$$

$$\begin{aligned} D = & -\frac{\sqrt{2}}{32\rho_{01}} \left(5 + \frac{27\tilde{\nu}^2 - 6\tilde{\nu} - 5}{\rho_{01}^4} \right) \xi^3 \cos \alpha (4 \cos^2 \alpha - 3) \\ & + \frac{3\sqrt{2}\iota \text{sign}(J)}{2\rho_{01}^3} \xi^2 \sin \alpha (4 \cos^2 \alpha - 1) \frac{\partial}{\partial \xi} \\ & + \frac{3\sqrt{2}\iota \text{sign}(J)}{2\rho_{01}^3} \xi \cos \alpha (4 \cos^2 \alpha - 3) \frac{\partial}{\partial \alpha}. \end{aligned} \quad (47)$$

Polar coordinates have been introduced according to $\xi^2 = y_m^2 + z_m^2$, $\tan \alpha = z_m/y_m$. The only nonvanishing matrix elements entering the sum (41) are the following

$$\langle 0, 0 | A | 0, 0 \rangle = -\frac{\sqrt{2}}{\rho_{01}}, \quad (48)$$

$$\langle 0, 0 | A | 0, \pm 2 \rangle = \pm \frac{3 \text{sign}(J)\tilde{\nu}}{2\rho_{01}^3\omega_2}, \quad (49)$$

$$\langle 0, \pm 2 | A | 0, 0 \rangle = \mp \frac{3 \text{sign}(J)\tilde{\nu}}{2\rho_{01}^3\omega_2}, \quad (50)$$

$$\langle 0, 0 | B | 0, 0 \rangle = -\frac{\sqrt{2}\omega_2}{2\rho_{01}}, \quad (51)$$

$$\begin{aligned} \langle 0, \pm 2 | B | 0, 0 \rangle = & \mp \text{sign}(J) \frac{3\tilde{\nu} + 2}{2\rho_{01}^3} - \frac{1}{4} \frac{\omega_2}{\rho_{01}} \\ & + \frac{9\tilde{\nu}^2 - 1}{4} \frac{1}{\omega_2\rho_{01}^5}, \end{aligned} \quad (52)$$

$$\begin{aligned} \langle 0, \mp 3 | D | 0, 0 \rangle = & -\frac{\sqrt{6}}{16\rho_{01}^5\omega_2^{3/2}} (5\rho_{01}^4 - 5 \\ & \mp 24\rho_{01}^2 \text{sign}(J)\tilde{\nu}\omega_2 + 27\tilde{\nu}^2 - 6\tilde{\nu}). \end{aligned} \quad (53)$$

Collecting everything, we arrive to

$$\begin{aligned}
\epsilon_4 = & \langle J, 0, 0, 0 | h'_4 | J, 0, 0, 0 \rangle - \left\{ \frac{\langle 0, 3 | D | 0, 0 \rangle^2}{3\omega_2 + 3\omega_3 \text{sign}(J)} \right. \\
& + \frac{\langle 0, -3 | D | 0, 0 \rangle^2}{3\omega_2 - 3\omega_3 \text{sign}(J)} \left. \right\} - \frac{11}{\omega_1^4} \langle 0, 0 | C | 0, 0 \rangle^2 \\
& + \frac{\omega_1}{4} \left\{ \frac{\langle 0, 0 | A | 0, 0 \rangle^2}{\omega_1} + \frac{\langle 0, 0 | A | 0, 2 \rangle \langle 0, 2 | A | 0, 0 \rangle}{\omega_1 + 2\omega_2 + 2\omega_3 \text{sign}(J)} \right. \\
& + \frac{\langle 0, 0 | A | 0, -2 \rangle \langle 0, -2 | A | 0, 0 \rangle}{\omega_1 + 2\omega_2 - 2\omega_3 \text{sign}(J)} \left. \right\} \\
& - \left\{ \frac{\langle 0, 0 | A | 0, 2 \rangle \langle 0, 2 | B | 0, 0 \rangle}{\omega_1 + 2\omega_2 + 2\omega_3 \text{sign}(J)} \right. \\
& + \frac{\langle 0, 0 | A | 0, -2 \rangle \langle 0, -2 | B | 0, 0 \rangle}{\omega_1 + 2\omega_2 - 2\omega_3 \text{sign}(J)} \left. \right\} \\
& - \frac{6}{\omega_1^3} \langle 0, 0 | B | 0, 0 \rangle \langle 0, 0 | C | 0, 0 \rangle \\
& - \frac{1}{\omega_1} \left\{ \frac{\langle 0, 0 | B | 0, 0 \rangle^2}{\omega_1} + \frac{\langle 0, 2 | B | 0, 0 \rangle^2}{\omega_1 + 2\omega_2 + 2\omega_3 \text{sign}(J)} \right. \\
& + \frac{\langle 0, -2 | B | 0, 0 \rangle^2}{\omega_1 + 2\omega_2 - 2\omega_3 \text{sign}(J)} \left. \right\}. \tag{54}
\end{aligned}$$

It may be checked that the corrections go to zero in both the Wigner ($\beta \rightarrow \infty$) and the oscillator ($\beta \rightarrow 0$) limits.

We show in Fig. 3 the relative weight of ϵ_4 in ϵ for three semions in states with $J = 3$ and $J = 6$. The numbers are similar to those appearing in the two-anyon problem. Thus, we expect a similar accuracy to this order, i.e. one part in 10^3 or better.

In Fig. 4, the levels with $J = \pm 6$ are drawn. β is increased from 0.5 to 8. Notice that the coulomb effects are stronger for the state with negative J , and that the levels become flatter (as a function of ν) as β rises.

In conclusion, the energy levels of two and three anyons in a model parabolic dot were computed by means of the $1/|J|$ -expansion. The qualitative picture emerging from the $1/|J|$ -expansion is that of a rigid structure (an orbit in the two-anyon system, an equilateral triangle for three anyons) against which harmonic and anharmonic oscillations are developed. The coulomb repulsion is much stronger for negative- J states. Comparison with exact particular solutions for two anyons shows excellent agreement.

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FIG. 1. Relative weight of ϵ_6 in ϵ . Two anyons in states with $n = 0$ and $\nu = 1/2$ are studied. a) $J = 2$, b) $J = 6$.

FIG. 2. Comparison between the $1/|J|$ -estimate and the exact solution found in [16] for two anyons with $J = 6$, $n = 0$.

FIG. 3. Relative weight of ϵ_4 for three anyons in states with $\nu = 1/2$. a) $J = 3$, b) $J = 6$.

FIG. 4. $|J|\epsilon$ vs ν for three anyons in states with $J = \pm 6$. a) $\beta = 0.5$, b) $\beta = 8$.

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